# UNIFORM STRUCTURES AND SQUARE ROOTS IN TOPOLOGICAL GROUPS

BY

P. ENFLO

## PART I.

ABSTRACT

Topological groups which are free from small subgroups and topological groups with locally uniformly continuous group multiplication are considered. Results concerning square roots, one-parameter subgroups and extensions of local groups are obtained as well as some generalisations of theorems for locally compact groups.

Our aim in this paper is to investigate, in the spirit of Hilbert's fifth problem, topological groups which are not locally compact. We shall approach the problem in two ways: the more abstract in Part I and the more concrete, for commutative groups, in Part II. Since the two approaches are interrelated we have brought them together in the same paper. In Enflo [1] we proved that locally Banach local groups, in which $x \to xy$ is continuously Frechet differentiable and $(x, y) \to xy$ is locally uniformly continuous, are analytic local groups, thus generalising the results of Segal [4] and Birkhoff [5]. In [1] it turned out that the local uniform continuity of the group multiplication is an essential assumption and so we will begin this investigation by studying topological groups with locally uniformly continuous group multiplication.

In Chapter 1 we derive some simple properties of topological groups with locally uniformly continuous group multiplication and also generalise some results for locally compact groups.

In Chapter 2 we study topological groups which are free from small subgroups and give theorems concerning uniqueness of square roots, the connection between square roots and one-parameter subgroups and so on. We also give some results

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concerning extensions of local groups and embeddings of commutative groups in metric linear spaces. The results from this chapter are needed in the last chapter of Part II of this paper. The results from the first two chapters often throw light on phenomena appearing in locally compact groups and we have pointed this out.

In Chapter 2 we introduce the concept "uniformly free from small subgroups" (which for locally compact groups is trivially the same as "free from small subgroups"). This turns out to be a fruitful concept, it will also be used frequently in Part II. Subsequently some ways to continue the study of Hilbert's fifth problem become clear. One is to extend differential calculus and Lie theory to the class of topological linear spaces which, regarded as topological groups under addition, are uniformly free from small subgroups. Another way is to characterise Banach spaces in terms of topological groups and so to study topological groups which have some of the properties which are found. In Chapter 3 we briefly discuss these approaches and also present some of the difficulties which we will encounter. In Part II we turn instead to commutative groups and make the assumption directly that the uniform structure of the group is the same as the uniform structure of some appropriate Banach space (or some similar assumption). With the aid of information we have from Part I this will give us in many cases quite good information as to the structure of the topological group.

In Chapter 4 (the first chapter of Part II) we give the necessary geometrical background for this approach.

In Chapter 5 we study uniformly Banach commutative groups with some Lipschitz condition on the group multiplication or the condition of uniform continuity only. We give theorems concerning existence and largeness of square roots. The results have immediate application to purely geometrical problems and so we have given the simplest of these as corollaries to the theorems. In the final section of this chapter we study subgroups of the additive groups of Banach spaces and give sufficient conditions for these to be linear subspaces. An example shows that Theorem 5.2.1 is in a sense the best possible.

However, it turns out in Chapter 5 that we must study purely geometrical problems in topological linear spaces to get more precise information of the group structure. This is done in Chapter 6. In this chapter we present, among other things, a beginning of a solution of the problem of classifying topological linear spaces from the point of view of uniform equivalence. This problem was also posed by

Bessaga [6] and Lindenstrauss [7]. For instance we prove that if a topological linear space which has a bounded neighbourhood of 0 is uniformly homeomorphic to a Hilbert space, then it is linearly isomorphic to the Hilbert space.

In Chapter 7 we combine results from the preceding chapters in order to obtain theorems concerning the structure of commutative groups.

Statements of some open problems occur throughout the paper.

## 1. Topological groups with locally uniformly continuous group multiplication.

### 1.1. Definitions and elementary properties of uniformities in topological groups.

For the definition of uniform space the reader is referred to Kelley [8]. We recall that if $G$ is a topological group, then the left uniformity for $G$ is the uniformity which has as a base the family of sets $\{(x, y) \mid x^{-1}y \in U\}$ where $U$ runs through the neighbourhoods of $e$ in $G$. The right uniformity is defined in the same way but we consider $xy^{-1}$ instead of $x^{-1}y$. In this paper we often talk about subgroups of linear spaces and there we always mean subgroups of the additive groups of the linear spaces. Thus topological linear spaces will be regarded as special types of topogical groups.

When we talk of a commutative group as a uniform space we will always mean the group with the left (or, equivalently, the right) uniformity. When we talk of a commutative metric group as a metric space we will always mean the group with some invariant metric. In this paper we will always assume that the topological groups are Hausdorff. We will always consider real linear spaces. A neighbourhood need not necessarily be an open set.

Let $X$ and $Y$ be uniform spaces and let $f$ be a map $X \to Y$. We shall say that a subset $A$ of $X$ is a uniform set for $f$ if for every member of the uniformity for $Y$ there is a member $U$ of the uniformity for $X$ such that $x \in A$ and $(x, y) \in U \Rightarrow (f(x), f(y)) \in V$.

PROPOSITION 1.1.1. *If $f$ is continuous and $A$ is compact, then $A$ is a uniform set for $f$.*

We omit the simple proof. The proposition shows that the concept "uniform set" in some sense generalises the concept "compact set".

PROPOSITION 1.1.2. *If, for some uniformity $\mathscr{U}$ for the topological group $G$, $(x, y) \to xy$ is uniformly continuous as a function $U^2 \times U^2 \to U^4$ for some symmetric neighbourhood $U$ of $e$, then $\mathscr{U}$ coincides on $U$ with the left and the* *right uniformity for G. The mapping $x \to x^{-1}$ is then uniformly continuous on U.*

PROOF. $\{(x^{-1}, x) \mid x \in U\}$ is a uniform set for $(x, y) \to xy$ with respect to $\mathscr{U}$. Thus for every member $P_1$ of the restriction of $\mathscr{U}$ to $U$ there is a member $P_2$ of the restriction of $\mathscr{U}$ to $U$ such that $(x, y) \in P_2 \Rightarrow (x^{-1}x, x^{-1}y) \in P_1$. Thus $(x, y) \in P_2 \Rightarrow (e, x^{-1}y) \in P_1$ and since $\mathscr{U}$ gives the topology of $G$, every member of the restriction of the left uniformity for $G$ to $U$ contains a member $P_2$ of the restriction of $\mathscr{U}$ to $U$.

On the other hand $\{(x, e) \mid x \in U\}$ is a uniform set for $(x, y) \to xy$ with respect to $\mathscr{U}$. Thus for every member $P_1$ of the restriction of $\mathscr{U}$ to $U$ there is a member $P_2$ of the restriction of $\mathscr{U}$ to $U$ such that $(e, x^{-1}y) \in P_2 \Rightarrow (x \cdot e, x \cdot x^{-1}y) \in P_1$, that is $(x, y) \in P_1$. Thus if $x^{-1}y$ is sufficiently close to $e$ then $(x, y) \in P_1$ and so $P_1$ contains a member of the restriction to $U$ of the left uniformity for $G$. The arguments above also hold for the right uniformity for $G$. Hence the left and the right uniformities for $G$ coincide on $U$ and $x \to x^{-1}$ is uniformly continuous on $U$. The proposition is proved.

Proposition 1.1.2 makes the following definitions natural. We say that a topological group $G$ is a locally uniform group if there is a uniform structure $\mathscr{U}$ for $G$ and a neighbourhood $U$ of $e$ in $G$ such that $(x, y) \to xy$ is uniformly continuous on $U \times U$ with respect to $\mathscr{U}$. We say that $G$ is a uniform group if $U$ can be chosen to be all of $G$.

PROPOSITION 1.1.3. *A topological group is a locally uniform group if and only if its left and its right uniformity coincide on some neighbourhood of e.*

PROOF. The "only if" part follows from Proposition 1.1.2. If the left and the right uniformity coincide on some neighbourhood of $e$ then there exists a symmetric neighbourhood $U$ of $e$ such that for every neighbourhood $V$ of $e$ there exists a neighbourhood $V_1$ with the property: $x \in U^3$, $y \in U^3$ and $x^{-1}y \in V_1 \Rightarrow xy^{-1} \in V$. Let $V$ be a neighbourhood of $e$ and let $V_1$ have the property described above. Let $V_2$ be a neighbourhood of $e$ such that $V_2^2 \subset V_1$ and let $V_3, V_3 \subset V_2$ be a symmetric neighbourhood of $e$ such that $x \in U^3$, $y \in U^3$ and $x^{-1}y \in V_3 \Rightarrow xy^{-1} \in V_2$. Now let $x, y, x_1, y_1$ be elements in $U$ and let $x_1^{-1}x \in V_3$ and $y_1^{-1}y \in V_3$ hold. Then $y_1y^{-1} \in V_2$ and $x^{-1}x_1 \in V_2$ and thus $y_1(y^{-1}x^{-1}x_1) \in V_1$. This gives that, $y_1^{-1}x_1^{-1}xy \in V$ and thus the group multiplication is locally uniformly continuous in the left uniformity of the group. The proposition is proved.

The proofs of Propositions 1.1.2 and 1.1.3 also give that $G$ is a uniform group if and only if the left and the right uniformity for $G$ coincide. The remaining propositions of this section give information of the behaviour of product groups and quotient groups.

PROPOSITION 1.1.4. *The direct product of a family of uniform groups is a uniform group.*

PROOF. Let $U$ be a neighbourhood of the unit element in the product group $G$. Then $U$ contains a neighbourhood $V$ of the form $V_{\alpha_1} \times V_{\alpha_2} \times \cdots \times V_{\alpha_n} \times G'_{\alpha_1\alpha_2\cdots\alpha_n}$ where $V_{\alpha_j}$ is a neighbourhood of $e$ in the group with index $\alpha_j$ in the product and $G'_{\alpha_1\alpha_2\ldots\alpha_n}$ is the direct product of the groups with indices different from $\alpha_1, \alpha_2, \cdots \alpha_n$. Now let $V_1 = V_{1,\alpha 2} \times V_{1,\alpha_2} \times \cdots \times V_{1,\alpha_n} \times G'_{\alpha_1\alpha_2\ldots\alpha_n}$ be a neighbourhood of $e$ in $G$, $V_{1,\alpha_j} \subset V_{\alpha_j}$ such that $x_{\alpha_j}^{-1} \cdot y_{\alpha_j} \in V_{1,\alpha_j} \Rightarrow x_{\alpha_j} \cdot y_{\alpha_j}^{-1} \in V_{\alpha_j}$. Then, in $G$, $x^{-1}y \in V_1 \Rightarrow xy^{-1} \in V \subset U$, and thus the left and the right uniformities for $G$ coincide. The remark following Proposition 1.1.3 completes the proof.

PROPOSITION 1.1.5. *The direct product of a family of locally uniform groups is locally uniform group if and only if all groups of the family are locally uniform and all but a finite number are uniform.*

PROOF. We begin with the "if" part. Let (with the same notation as above) $V = V_{\alpha_1} \times V_{\alpha_2 \ldots} \times V_{\alpha_n} \times G'_{\alpha_1\alpha_2\ldots\alpha_n}$ be a neighbourhood of $e$ in $G$ such that $G'_{\alpha_1\alpha_2\ldots\alpha_n}$ is uniform and the left and the right uniformities for the group with index $\alpha_j$ coincide on $V_{\alpha_j}$. If $U$ is any neighbourhood of $e$ in $G$ then $U$ contains a neighbourhood $V_1$ of $e$, $V_1 \subset V$ of the form $V_1 = V_{1,\alpha_1} \times V_{1,\alpha_2\ldots} \times V_{1,\alpha_n} \times V_{1,\alpha_{n+1}} \cdots \times V_{1,\alpha_m} \times G'_{\alpha_1\alpha_2\ldots\alpha_m}$ and $V_1$ contains a neighbourhood $V_2$ of $e$

$$V_2 = V_{2,\alpha_1} \times V_{2,\alpha_2} \cdots \times V_{2,\alpha_m} \times G'_{\alpha_1\alpha_2\ldots\alpha_m}$$

such that $x_{\alpha_j} \in V_{\alpha_j}$, $y_{\alpha_j} \in V_{\alpha_j}$ and $x_{\alpha_j}^{-1} \cdot y_{\alpha_j} \in V_{2,\alpha_j} \Rightarrow x_{\alpha_j} \cdot y_{\alpha_j} \in V_{1,\alpha_j}^{-1}$, $1 \leqq j \leqq m$ (for $n < j \leqq m$, $V_{\alpha_j}$ is the whole group with index $\alpha_j$). Thus $x \in V$, $y \in V$ and $x^{-1}y \in V_2 \Rightarrow xy^{-1} \in V_1 \subset U$. This gives that the left and the right uniformities for $G$ are locally the same and proposition 1.1.3 gives that $G$ is locally uniform.

If in the direct product $G$ there is an infinite number of groups which are not uniform so that, if $V$ is a neighbourhood of $e$ in $G$, $V$ contains of group of the form $(e, e, \cdots, e, G_{\alpha_1}, e, \cdots)$ where $G_{\alpha_1}$ is not uniform. Thus in $G_{\alpha_1}$ there is a neighbourhood $V_{1,\alpha_1}$ of $e$ such that for every neighbourhood $V_{2,\alpha_1}$ of $e$ there are elements $x_{\alpha_1} \in G_{\alpha_1}$, $y_{\alpha_1} \in G_{\alpha_1}$ such that $x_{\alpha_1}^{-1} \cdot y_{\alpha_1} \in V_{2,\alpha_1}$ and $x_{\alpha_1} \cdot y_{\alpha_1}^{-1} \notin V_{1,\alpha_1}$. Thus if $V_1$ is the neighbourhood $G'_{\alpha_1} \times V_{1,\alpha_1}$ of $e$ in $G$ then for every neighbourhood $V_2$ of $e$ in $G$ there are $x \in V$ and $y \in V$ such that $x^{-1}y \in V_2$ but $xy^{-1} \notin V_1$. Thus the left and the right uniformity for $G$ do not coincide on $V$ and so $G$ is not locally uniform. The proposition is proved.

PROPOSITION 1.1.6. *If $H$ is a closed normal subgroup of the locally uniform group $G$, then $G/H$ is a locally uniform group.*

PROOF. Let $T$ be the natural mapping $G \to G/H$. Then $T$ is open and continuous. Let $U$ be a neighbourhood of $e$ in $G$ such that the left and the right uniformities for $G$ coincide on $U^2$. We prove that the left and the right uniformities for $G/H$ coincide on $T(U)$. Let $V$ be a neighbourhood of $e$ in $G$, $V \subset U$, and let $W$ be a neighbourhood of $e$ in $G$, $W \subset U$, such that $x \in U^2$, $y \in U^2$ and $x^{-1}y \in W \Rightarrow xy^{-1} \in V$. Now let $p_1 \in T(U)$, $p_2 \in T(U)$ and $p_1^{-1}p_2 \in T(W)$ hold. Then $p_1^{-1}p_2 = T(y)$ for some $y \in W$ and if we put $p_1 = T(x)$ where $x \in U$ then we get $p_2 = T(xy)$ where $xy \in U^2$. Thus $p_1p_2^{-1} = T(xy^{-1}x^{-1})$ and since $x^{-1}(xy) \in W$ we have $xy^{-1}x^{-1} \in V$ and $p_1p_2^{-1} \in T(V)$. Thus the left and the right uniformities for $G/H$ coincide on $T(U)$ and Proposition 1.1.3 gives that $G/H$ is a locally uniform group.

The proof of Proposition 1.1.6 also gives that $G/H$ is uniform if $G$ is uniform.

If $G$ is a topological group, $H$ a closed subgroup of $G$ and if $H$ and the left coset space $G/H$ are locally compact, then $G$ is locally compact (see Montgomery and Zippin [9] pp. 52–53). There is no similar theorem for locally uniform groups. In fact, in Example 2.2.2 below, $H$ and $G/H$ are both commutative (and hence uniform) but $G$ is not uniform. By taking a suitable product of such examples we obtain an example where $G$ is not even locally uniform.

LEMMA. *$G$ is a uniform group if and only if for every neighbourhood $U$ of $e$ there is a neighbourhhod $V$ of $e$ such that for all $x \in G$, $x^{-1}Vx \subset U$.*

We omit the simple proof.

PROPOSITION 1.1.7. *If $G$ is a connected and locally connected topological group, $H$ is a discrete normal subgroup of $G$ and $G/H$ is uniform, then $G$ is uniform.*

PROOF. Let $U$ be a symmetric neighbourhood of $e$ in $G$ such that $U^2 \cap H = \{e\}$. Let $T$ be the natural map $G \to G/H$. Let $V$ be a connected neighbourhood of $e$ in $G$, $V \subset U$ such that for every $p \in G/H$ $p^{-1}T(V)p \subset T(U)$, by the preceding lemma such a $V$ exists. Then for every $x \in G$ we have $T(x^{-1}Vx) \in T(U)$ and since $x^{-1}Vx$

is connected and $U^2 \cap H = \{e\}$ we have $x^{-1}Vx \subset U$. The preceding lemma completes the proof.

It follows from Proposition 1.1.7 that the universal covering group of a compact connected Lie-group is uniform and so by Proposition 1.1.6 every connected Lie-group which is locally isomorphic to a compact group is a uniform group. This observation suggests the problem of extending results proved for connected Lie-groups locally isomorphic to compact groups to uniform, connected, locally Banach analytical groups. For instance, it has been proved that in a connected Lie-group which is locally isomorphic to a compact group every element lies on a one-parameter subgroup (see Tondeur [10]) and it seems very likely that this conclusion holds for a much larger class of uniform, analytical groups.

**1.2. Transformation groups.** We shall say that a topological group $G$ is locally generated, if for every neighbourhood $V$ of $e$ the smallest subgroup of $G$ which contains $V$ is $G$ itself. The reason for introducing this concept is that theorems which are proved for connected groups are in fact often proved automatically for locally generated groups and we shall in this paper give an example of a complete, commutative locally generated group which has more than one element and is totally disconnected, thus showing that there may be an essential difference between the concepts (Example 2.2.1). We observe that if $H$ is an open and closed subgroup of a locally generated group $G$, then $H = G$.

THEOREM 1.2.1. *If a locally generated, uniform group acts effectively as a transformation group on the real line, then it is commutative.*

PROOF. Since the group is locally generated all transformations of the real line are monotone, increasing functions. This shows that they can be regarded as transformations of the extended real line with $+\infty$ and $-\infty$ as fixed points and we will regard them as such. We now show that if $M_f$ is the set of fixed points for $f$ then the points in $M_f - \dot{M}_f$ are fixed points for all transformations of the group.

If $x \in M_f - \dot{M}_f$ then in every neighbourhood of $x$ there is a point $x_1$ (which may be the same as $x$) such that $f(x_1) = x_1$ and $f(y) \neq y$ for all $y$ in some interval to the right (or to the left) of $x_1$. We can assume $f(y) > y$ for all $y$ in an interval to the right of $x_1$ (otherwise consider $f^{-1}$). Assume that there is a transformation $g \in G$ with $g(x_1) \neq x_1$. Then, in every neighbourhood of $e$ in $G$ there is an element $h$ with $h(x_1) \neq x_1$ since $G$ is locally generated. We assume $h(x_1) > x_1$ (otherwise consider $h^{-1}$). Now since $G$ is uniform $f^n h f^{-n}$ is near $e$ for all $n$ if $h$ is near $e$. But as $n$ tends to infinity $f^n h f^{-n}(x_1)$ tends to a fixed point of $f$ to to the right of $x_1$. This is a contradiction and shows that $x_1$ is a fixed point for all transformations of $G$ and thus all elements of $M_f - \dot{M}_f$ are fixed points for all trans formations of $G$.

The discussion above shows that it is enough to prove that any two transformations of $G$ commute in every interval $[a, b]$ which has the property that for all $f \in G$ we have $1/f(a) = a$, $f(b) = b$, $2/f(x) = x$ for all $x$ in $(a, b)$ or $f(x) \neq x$ for all $x$ in $(a, b)$. We consider such an interval. If $f(x) > g(x)$ for some $x$ in $(a, b)$ then

$$f(x) > g(x) \text{ for all } x \text{ in } (a, b). \tag{1}$$

Otherwise, $f^{-1}g$ would have a fixed point in $(a, b)$ without having all points in $(a, b)$ as fixed points. A net $g_n$ such that $g_n \to e$ has the property that $g_n(x) \to x$ for all $x$ in $(a, b)$. This and (1) gives that if for a sequence $f_n$, $f_n(x) \to x$ for some $x$ in $(a, b)$ then $f_n(y) \to y$ for all $y$ in $(a, b)$ and (1) and Dini's theorem give that the convergence is uniform. For every transformation $g \in G$ and every $x \in (a, b)$ we form the set $N_{g,x} = \{g^n(x) \mid n = 0, \pm 1, \pm 2, \cdots\}$. If $\varepsilon > 0$ then by choosing $g$ sufficiently near $e$ we obtain

$$\sup_{x, y \in (a,b)} d(y, N_{g,x}) < \varepsilon. \tag{2}$$

Now if for some $x \in (a, b)$ we have $h(x) = g^m(x)$ and $f(x) = g^n(x)$ then $h$ and $f$ commute on $(a, b)$. This fact and (2) give that if $h$ and $f$ are transformations in $G$ we can find sequences $h_n$ and $f_n$ such that $h_n$ and $f_n$ commute on $(a, b)$ and $h_n(x) \to h(x)$ and $f_n(x) \to f(x)$ for some $x$ in $(a, b)$. And then $h_n \to h$ and $f_n \to f$ uniformly on $(a, b)$ and thus $h$ and $f$ commute on $(a, b)$. This completes the proof of the theorem.

The proof of Theorem 1.2.1 gives, in fact, in fact, more information on the group structure than that it is commutative. However, it seems that little can be said about the topology of $G$ if only we make the assumption that it is locally generated and uniform. In this paper we will go no further into the large field of transformation groups.

**1.3. Locally uniformly open groups.** In this section we generalise some of the results of [9], pp. 54–56. We shall say that a set $M$ in a topological group is uniformly open if there is a neighbourhood $V$ of $e$ such that $x \in M \Rightarrow Vx \subset M$. An open subgroup of a topological group is obviously uniformly open. The following theorem gives information on the structure of uniformly open sets.

THEOREM 1.3.1. *A set $M$ in a topological group is uniformly open if and only if it is a union of right cosets of an open subgroup.*

PROOF. The "if" part is quite obvious and so we assume that $M$ is a uniformly open set. Let $V$ be a symmetric neighbourhood of $e$ such that $x \in M \Rightarrow Vx \subset M$. Thus if $U \subset M$, then $VU \subset M$. And by induction on $n$ this gives that if $x \in M$ then $V^n x \subset M$ and so $\bigcup_{n=1}^{\infty} V^n)x \subset M$. This gives that $M$ is a union of right cosets of the open subgroup $\bigcup_{n=1}^{\infty} V^n$. We immediately get

COROLLARY 1. *Let $G$ be a topological group and let $U$ be a uniformly open neighbourhood of $e$. Then $U$ contains an open subgroup.*

The concept "uniformly open set" generalises the concept "compact open set" since a compact open set obviously is uniformly open. We shall say that a topological group is locally uniformly open if there is a base for the neighbourhood system of $e$ which consists of uniformly open neighbourhoods. A locally uniformly open group is totally disconnected. The product of a family of totally disconnected locally compact groups is locally uniformly open. A complete totally disconnected group need not be locally uniformly open as is shown by Example 1.3.1 below. Theorem 1.3.1 immediately gives

COROLLARY 2. *Let $G$ be a locally uniformly open topological group and let $V$ be a neighbourhood of $e$ in $G$. Then $V$ contains a uniformly open subgroup.*

We now turn to uniform groups.

LEMMA. *Let $G$ be a uniform group and let $U$ be a neighbourhood of $e$. There then exists a neighbourhood $V$ of $e$ such that $V \subset x^{-1}Ux$ for all $x \in G$.*

PROOF. Since $G$ is uniform there exists a neighbourhood $V$ of $e$ such that $xVx^{-1} \subset U$ for all $x \in G$. By multiplying the last inclusion to the left by $x^{-1}$ and to the right by $x$ we get the desired result.

THEOREM 1.3.2. *If $G$ is a locally uniformly open uniform group and $U$ is a neighbourhood of $e$ in $G$, then $U$ contains a uniformly open invariant subgroup $H$. $G/H$ is then a discrete group.*

PROOF. The Corollary 2 of Theorem 1.3.1 gives that $U$ contains a uniformly open subgroup, call it $H'$. Put $H = \bigcap_{x \in G} x^{-1}H'x$. Then $H$ is by definition an invariant subgroup. The preceding lemma shows that it is open and hence uniformly open. Since $H$ is open $G/H$ is discrete.

Reference [9] p. 57 gives an example of a totally disconnected locally compact group which has no invariant compact open subgroups. Theorem 1.3.2 shows that such a group cannot be uniform.

EXAMPLE 1.3.1. Let $G_n$ be the group $\{e^{2\pi ik/n}\ , k = 0, 1, \cdots, n-1\}$ and let $G$ be algebraically the product of the $G_n$. Define a metric in $G$ by putting $d(a,b) = = \sup d(a_n, b_n)$. Then $G$ becomes a complete, metric, totally disconnected, commutative group. But $G$ is not locally uniformly open.

A locally uniformly open group with more than one element is not locally generated, by Corollary 1 of Theorem 1.3.1. The group in Example 1.3.1 is not locally generated. However, in Example 2.2.1 below, we give an example of a locally generated, complete, metric, totally disconnected, commutative group. It seems obvious to us that by a method similar to that used in example 2.2.1, it is possible to construct a totally disconnected, locally generated, complete subgroup of the Hilbert space.

## 2. Groups which are free from small subgroups.

**2.1. Groups which are free from small subgroups and locally uniform groups.** We shall say that a topological group is free from small subgroups if there is a neighbourhood $U$ of $e$ such that $x \neq e \Rightarrow x^n \notin U$ for some $n$. We say that such an $U$ is free from subgroups. We shall say that a topological group is dissipative if there is a neighbourhood $U$ of $e$ such that for every $x \neq e$ there is an $n_x$ such that $x^n \notin U$ if $n \geqq n_x$. We shall say that $U$ is a dissipative neighbourhood of $e$. A dissipative group is obviously free from small subgroups but the converse is not true. We shall say that a topological group is uniformly free from small subgroups if there is a neighbourhood $U$ of $e$ such that for every neighbourhood $V$ of $e$ there is a number $n_V$ such that $x \notin V \Rightarrow x^n \notin U$ for some $n \leqq n_V$. We say that such a $U$ is uniformly free from subgroups. We say that a topological group is uniformly dissipative if there is a neighbourhood $U$ of $e$ such that for every neighbourhood $V$ of $e$ there is a number $n_V$ such that $x \notin V \Rightarrow x^n \notin U$ for all $n \geqq n_V$. We say that such a $U$ is a uniformly dissipative neighbourhood of $e$. A uniformly dissipative group is obviously uniformly free from small subgroups, but the converse is not true. In Chapter 3 we show that for neighbourhoods of 0 in topological linear spaces the concepts "uniformly dissipative" and "bounded" coincide. It is easy to see that a locally compact group which is free from small subgroups is uniformly free from small subgroups and that a dissipative locally compact group is uniformly dissipative. Example 2.1.1 and Theorem 2.1.1 show that this is not true for locally uniform groups.

EXAMPLE 2.1.1. Let $G$ be algebraically a countable infinite product of real lines. Let a base for the topology in $G$ be the sets which are products of open intervals. $G$ then becomes a topological group. Since no base at $e$ is countable, $G$ is non-metrizable. Since $G$ is free from small subgroups it cannot be approximated by metric groups (in the sense that every neighbourhood of $e$ contains a subgroup $H$ such that $G/H$ is metrizable) in contrast to the theorem on page 58 in [9].

Non-existence of small subgroups does not imply that a topological group is a locally uniform group, in contrast to Theorem 2.1.1. This is shown by Example 2 and Theorem 3.3 in [1].

THEOREM 2.1.1. *If a topological group is uniformly free from small subgroups then it is a metrizable, locally uniform group.*

PROOF. Let $U$ be a neighbourhood of $e$ which is uniformly free from subgroups, and let $W_n$ be a sequence of neighbourhoods of $e$ such that for every $n$, $\bigcup_{k=1}^{n} W_n^k \subset U$. Then $\{W_n\}$ is a base for the neighbourhoods of $e$. Let $V$ be a neighbourhood of $e$, then if $x \notin V$ we have $x^n \notin U$ for some $n \leqq n_V$ and this implies that $x \notin W_{n_V}$. Thus $W_{n_V} \subset V$ and this gives that $G$ is metrizable. Let $d$ be a right invariant metric in $G$, i.e. $d(xa, ya) = d(x, y)$. Then the uniformity of $d$ is the right uniformity of $G$. Let $\{x \mid d(x,e) \leqq r\}$ be a sphere in $U$. Assume that the group multiplication is not uniformly continuous in the sphere

$$\left\{(x,y) \,\middle|\, d(x,e) \leqq \frac{r}{8},\ d(y,e) \leqq \frac{r}{8}\right\}.$$

Then there is a positive $\varepsilon$ such that for every $\delta > 0$, $\delta < \varepsilon/2$, there are $x_0, y_0, x_1, y_1$, $d(x_j, e) \leqq r/8$, $d(y_j, e) \leqq r/8$, $d(x_0, x_1) < \delta$, $d(y_0, y_1) < \delta$, and $d(x_0 y_0, x_1 y_1) > \varepsilon$ Then $d(x_0 y_0, x_0 y_1) > \varepsilon - \delta > \varepsilon/2$ and $d(x_0^{-1} x_0 y_0, x_0^{-1} x_0 y_1) = d(y_0, y_1) < \delta$ that is, in the sphere $\{(x,y) \mid d(x,e) \leqq r/4,\ d(y,e) \leqq r/4\}$ there exist for every $\delta > 0$, $x_2 = x_0^{-1}$, $y_2 = x_0 y_0, y_3 = x_0 y_1$ such that $d(y_2, y_3) > \varepsilon/2$ and $d(x_2 y_2, x_2 y_3) < \delta$. Put $y_2 = w y_3$. Then $d(w,e) > \varepsilon/2$ and $d(x_2 w, x_2) = d(x_2 w y_3, x_2 y_3) = d(x_2 y_2, x_2 y_3) < \delta$. And then $d(x_2 w^n, x_2 w^{n-1}) = d(x_2 w w^{n-1}, x_2 w^{n-1}) < \delta$ and this gives $d(x_2 w^n, e)$ $< d(x_2, e) + n\delta \leqq r/8 + n\delta$. On the other hand, since $d(w,e) > \varepsilon/2$ there exists a number $N$ depending only on $\varepsilon$ such that for some $n \leqq N$ we have $d(w^n, e) > r$. But then $d(x_2 w^n, e) > r - r/8 = 7r/8$. Since $\delta > 0$ is arbitrary this is a contradiction and the theorem is proved.

Theorem 2.1.1 shows that it is natural to study locally uniform goups in connection with Lie theory. We notice that there is no corresponding theorem for locally compact groups since the conclusion is that the group is locally uniform. We also notice that the same discussion as in the proof of Theorem 2.1.1 will improve theorem 3.1 in [1]: A local $L$-group is an analytic local group if and only if it is uniformly free from small subgroups.

**2.2. Square roots and one-parameter subgroups.** We first recall the following theorem on uniqueness of square roots (see [1] and [9] p. 120).

THEOREM 2.2.1. *If a locally uniform group is free from small subgroups then there is a neighbourhood $U$ of $e$ such that if $x \in U$, $y \in U$ and $x^2 = y^2$, then $x = y$.*

In [1] it is shown that the theorem becomes false if the condition "locally uniform" is removed. The Theorems 2.1.1 and 2.2.1 immediately give

THEOREM 2.2.2. *If a topological group is uniformly free from small subgroups then there is a neighbourhood $U$ of $e$ such that if $x \in U$, $y \in U$ and $x^2 = y^2$, then $x = y$.*

In the following theorems of this section we consider the connection between square roots and one-parameter subgroups. If $G$ is a topological group we shall say that a subset $M$ of $G$ is a one-parameter subgroup if $M$ contains at least two points and there is a continuous map $t \to x(t)$ from the real line onto $M$ such that $x(t_1) \cdot x(t_2) = x(t_1 + t_2)$. We shall say that a topological group is locally complete if there is a neighbourhood of $e$ which is complete with respect to the left and the right uniformity of $G$. In the following theorem we will for simplicity use additive notation of the group operation.

THEOREM 2.2.3. *If $G$ is a commutative, uniformly dissipative, locally generated and locally complete group and the set of elements of the form $2x$ is dense in $G$, then $G$ is a topological linear space.*

PROOF. Since $G$ is metrizable by Theorem 2.1.1 we can assume that $d$ is an invariant metric in $G$. If $d(2x_n, 2y_n) \to 0$ for a sequence of pairs $(x_n, y_n)$ then $d(x_n, y_n) \to 0$. For otherwise, if we put $a_n = x_n - y_n$, some subsequence of $a_n$ would be bounded away from $e$ but $2ma_n$ would tend to $e$ as $n \to \infty$ for every fixed natural number $m$, and this contradicts the assumption that the group is uniformly dissipative. This gives in particular that $2x = 2y \Rightarrow x = y$ and that the well defined mapping $x \to x/2$ is a uniformly continuous isomorphism. Since the group is uniformly dissipative we have that if $W$ is a uniformly dissipative neighbourhood of $e$ then the solutions $x_n$ of the equations $2^n x_n = y$ tend to $e$ as $n \to \infty$ uniformly for $y \in W$. We consider such a $W$. We form the set

$$U_0 = \{2^{-n}y \mid n = 0,1,2,\cdots, y \in W\} \text{ and the sets } U_n = \{2^{-n}y \mid y \in U_0\}$$

Then the family of sets $U_n$ forms a fundamental system of neighbourhoods of $e$. Thus we have for some $k$, $U_k + U_k \subset U_0$ and this gives $U_{n+k} + U_{n+k} \subset U_n$ for every $n$.

Let $N$ be a natural number, $N > k$, and consider the sets $M_p = \{N + pk, N + pk + 1, N + pk + 2, \cdots, N + (p+1)k - 1\}$ where $p$ is a non-negative integer. Let $y \in U_0$. We shall say that a finite sum $\Sigma 2^{-m_j} \cdot y$ is of type $A$ if all $m_j$ are different and $> N$ and if for every $p$, $m_j \in M_p$ for at most one $j$. We have that if $z = \Sigma 2^{-m_j} \cdot y$ where the sum is of type $A$, then $z \in U_{N-k}$. For put $z = 2^{-m_0} \cdot y + 2^{-m_1} \cdot y + \cdots + 2^{-m_j} \cdot y$ where $m_r > m_{r-1}$. Then the sum of the last two terms is certainly an element of $U_{N+(j-1)k} + U_{N+jk} \subset U_{N+(j-2)k}$. Thus the sum of the last three terms is an element of $U_{N+(j-2)k} + U_{N+(j-2)k} \subset U_{N+(j-3)k}$. By repeating this discussion $j$ times we get $z \in U_{N-k}$. We have that every sum $\Sigma 2^{-m_j} \cdot y$ where all $m_j$ are different and all $m_j > N$ can be written as $k$ sums of type $A$ and thus every such sum is an element of $U_{N-k} + U_{N-k} + \cdots + U_{N-k}$ (with $k$ terms $U_{N-k}$). This gives that the mapping $\alpha \to \alpha y$ from the positive dyadic rationals into $G$ is uniformly continuous and so it can be uniquely extended to all positive real numbers and so it can be uniquely extended to all real numbers. Since the group is locally generated we can for all $z \in G$ uniquely extend the mapping $\alpha \to \alpha z$ from the positive dyadic rationals into $G$ to all real numbers $\alpha$. This defines $\alpha z$ for all real numbers $\alpha$ and all $z \in G$. The laws $\beta(\alpha z) = (\beta\alpha)z$ and the distributive laws of linear spaces hold since they hold for dyadic rationals and the discussion above and the formula $d(\alpha y, \alpha_0 y_0) \leqq d(\alpha y, \alpha y_0) + d(\alpha y_0, \alpha_0 y_0)$ gives that $(\alpha, y) \to \alpha y$ is continuous. The theorem is proved.

Example 2.2.1 below shows the importance of the condition "uniformly dissipative" in Theorem 2.3.3. Let $V$ be a symmetric neighbourhood of $e$ in a topological group $G$. We shall say that $x \in V$ lies on a unique local one-parameter subgroup in $V$ if there exists a unique function $\alpha \to x^\alpha$ from $[-1,1]$ into $V$ such that $x^{\alpha_1} \cdot x^{\alpha_2} = x^{\alpha_1 + \alpha_2}$.

THEOREM 2.2.4. *If $G$ is a uniformly dissipative, locally complete topological group such that for every neighbourhood $U$ of $e$ the set $\{x^2 \mid x \in U\}$ is dense in some neighbourhood of $e$, then there exists a symmetric neighbourhood $V$ of $e$ such that every element in $V$ lies on a unique local one-parameter subgroup in $V$.*

PROOF. It follows from Theorem 2.1.1 that $G$ is metrizable and locally uniform and we assume that $d$ is a right invariant metric in $G$. We choose an $\varepsilon > 0$ such that the $2\varepsilon$-sphere around $e$ is uniformly dissipative and such that group multiplication is uniformly continuous in the $3\varepsilon$-sphere around $e$. We choose a $\delta > 0$, $\delta < \varepsilon$, such that $d(x,e) < 2\varepsilon$ and $d(a,e) < \delta \Rightarrow d(a^{-1}xa, x) < \varepsilon$. We assume that there is a sequence of pairs $(x_n, y_n)$, $d(x_n, e) < \delta$, $d(y_n, e) < \delta$ such that $d(x_n^2, y_n^2) \to 0$ and $d(x_n^{-1} y_n, e) > \delta_1$ for all $n$ and some positive $\delta_1$. We put $a_n = x_n^{-1} y_n$. Then we have $d(a_n, e) < 2\varepsilon$ and thus we have $d(x_n^{-1} a_n x_n, a_n) < \varepsilon$. We also have $d(x_n^{-1} a_n x_n, a_n^{-1}) \to 0$ as $n \to \infty$. The locally uniformly continuous group multiplication gives easily that this implies that $d(x_n^{-1} \cdot a_n^{2^m} \cdot x_n, a^{-2^m}) \to 0$ for every natural number $m$. Thus, for every natural number $m$ and every positive $\delta_2 < \delta$ there exists a $w$ such that $d(x_n^{-1} \cdot a_n^{2} \cdot x_n, a_n^{-2^j}) < \delta_2$ for all $0 \leqq j \leqq m$ and all $n > w$. Then if $d(a_n^{2^j}, e) < 2\varepsilon, j \leqq m$, we have

$$d(a_n^{2^{j+j}}, e) \leqq d(a_n^{2^{j+j}}, x_n^{-1} \cdot a_n^{2^j} \cdot x_n \cdot a_n^{2^j}) + d(x_n^{-1} \cdot a_n^{2^j} \cdot x_n \cdot a_n^{2^j}, a_n^{-2^j} \cdot a_n^{2^j})$$

$$< \varepsilon + \delta_2$$

which contradicts that the $2\varepsilon$-sphere around $e$ is bounded. This gives that foı every neighbourhood $U$ of $e$ the set $\{x^2 \mid x \in U\}$ is a neighbourhood of $e$. Let $W$ be a uniformly dissipative neighbourhood of $e$ such that $x \notin W \Rightarrow x^{2n} \notin W$ for all $n \geqq n_W$ and such that every element of $W$ has a square root in $G$ and such that $x \in W$, $y \in W$ and $x^2 = y^2 \Rightarrow x = y$. Now if $V_1$ is a sufficiently small neighbourhood of $e$, then every element of $V_1$ has a unique $2^{n_W}$:th root in $W$ and thus for every $y \in V_1$ and every $n$ the equation $2^n x_n = y$ has a unique solution in $W$, and if $V_1$ is sufficiently small the $x_n$:s must satisfy $x_n = x_{n-1}^2$. For $y \in V_1$ consider the commutative group generated by the $x_n$:s. This group satisfies the conditions of Theorem 2.2.3 and thus we can uniquely extend the mapping $\alpha \to y^\alpha$ defined for dyadic rationals $\alpha$ to real $\alpha$:s. Now the set $V = \{y^\alpha \mid |\alpha| \leqq 1, y \in V_1\}$ is a symmetric neighbourhood of $e$ in which every element lies on a unique local one-parameter subgroup.

For the proof of the next theorem we will now introduce a concept which will be of frequent use in Part II of this paper. We shall say that a sequence of

$n+1$ points $y = x_0, x_1, \cdots, x_n = z$ in a metric space is an $\varepsilon$-chain between $y$ and $z$ if $d(x_i, x_{i+1}) \leqq \varepsilon$ for $0 \leqq i \leqq n-1$. We shall say that $\sum d(x_i, x_{i+1})$ is the length of the chain.

THEOREM 2.2.5. *If in a locally generated, locally complete topological group $G$, (1) there is a neighbourhood $U$ of $e$ such that $U^n$ is uniformly dissipative for all $n$, (2) the set of elements of the form $x^2$ is dense in $G$, then every element in $G$ lies on a unique one-parameter subgroup.*

PROOF. By Theorem 2.1.1 $G$ is locally uniform and metrizable. Let $d$ be a right invariant metric in $G$ and choose an $\varepsilon > 0$ such that the $\varepsilon$-sphere around $e$ is contained in $U$. Introduce a new metric $d_1$ in $G$ by letting $d_1(x,y)$ be the infimum of the lengths of the $\varepsilon$-chains between $x$ and $y$. Then the uniform structures of $d$ and $d_1$ are the same for $d(x,y) = d_1(x,y)$ if $d(x,y) \leqq \varepsilon$. We now assume that $G$ is given the metric $d_1$. Since $d_1$ is also right invariant the $n\varepsilon$-sphere around $e$ is contained in $U^n$. This gives that for every $\delta > 0$, $d_1(x^{2^n}, e) \to \infty$ as $n \to \infty$, uniformly for $x$ in the set $\{x \mid d_1(x,e) > \delta\}$. This implies also that $(*)$ $d_1(x^2, e) \to \infty$ as $d(x,e) \to \infty$. Assume that there is a sequence $x_n$ with $d_1(x_n, e) \to \infty$ but with $d_1(x_n^2, e) < w$ for all $n$ and some $w$. Since $x_n$ has an approximate $2^m$:*th* root for every $m$ and $d_1(z^{2^m}, e) \leqq 2^m \cdot d(z,e)$ for every $m$ and $z$ this would contradict that the $w$-sphere around $e$ is bounded. We now observe that $(x,y) \to xy$ is uniformly continuous in the sphere $d_1(x,e) \leqq w$, $d_1(y,e) \leqq w$ for every $w$. This follows from the proof of Theorem 2.1.1 where it is proved that $(x,y) \to xy$ is uniformly continuous in the sphere $d(x,e) \leqq r/8$, $d(y,e) \leqq r/8$ if the sphere $d(x,e) \leqq r$ is group-bounded. Let $x \in G$ and let $x_n$ be a sequence such that $x_n \to z$. Then $d_1(x_n, e)$ is a bounded sequence by $(*)$. The proof that $x_n$ is a Cauchy sequence is now exactly the same as in the proof of Theorem 2.2.4. This gives the existence and uniqueness of the solution of the equation $x^2 = z$ for every $z \in G$. Since every $w$-sphere around $e$ is bounded the solutions $x_n$ of the equations $x_n^{2^n} = z$ tend to $e$ as $n \to \infty$ uniformly in every sphere $d_1(z,e) < w$. Theorem 2.2.3 now completes the proof of Theorem 2.2.5 in exactly the same way as in Theorem 2.2.4.

Theorem 2.2.5 becomes false if "uniformly dissipative" is changed to "uniformly free from subgroup" in condition (1). This is shown by Example 2.2.2 below where neither the existence nor the uniqueness of square roots is true for all $z \in G$.

EXAMPLE 2.2.1. We consider the group $G$ of numbers of the form $\sum \pm 2^{m_j}$ where every sum is finite and the $m_j$: $s$ are integers. We now define a metric $d$ in $G$.

Put $d(2^n, 0) = 1$ if $n \geqq 0$ and $-(1/n)$ if $n < 0$. If $y$ and $z$ are both of the form $\Sigma \pm 2^{m_j}$ we shall say that a sequence of points $y = x_0, x_1 \cdots, x_n = z$ is a special chain between $y$ and $z$ if for $n - 1$, $|x_i - x_{i+1}| = 2^{m_i}$ for some integer $m_i$. We shall say that $\Sigma d(|x_i - x_{i+1}|, 0)$ is the length of the chain. We let $d(y, z)$ be the infimum of the lengths of the special chains between $y$ and $z$. It is obvious that this definition of $d(y, z)$ is consistent with the definition of $d(2^n, 0)$ and it is obvious that $d$ is an invariant metric. Thus $G$ is a topological group under addition. We have obviously $d(2x, 0) \geqq d(x, 0)$ and as $n \to \infty$ we have $\lim d(nx, 0) \geqq 1$. Thus in the completion $\bar{G}$ of $G$ we have a complete, locally generated, dissipative, commutative, metric group in which every element has a unique square root and in which for every $x = 0$, $d(nx, 0) \geqq 1$ for sufficiently large $n$. It is also a simple verification that if $2^n x_n = y$, then $x_n \to 0$ as $n \to \infty$. We now prove that $\bar{G}$ is totally disconnected in strong contrast to Theorem 2.2.3.

Every element of $\bar{G}$ corresponds in an obvious way to a real number, but the converse of this is not true. The sums $\Sigma_{n=k} 2^{-4n}$ do not correspond to elements of $\bar{G}$ for any $k \geqq 1$. For it is easy to see that the shortest chain between 0 and a finite partial sum $\Sigma_{n=k}^{m} 2^{-4n}$ has length $\Sigma_{n=k}^{m} 1/4n$ which tends to infinity with $m$. Now if $M$ is a subset of $\bar{G}$ which contains 0 and an element $x > 0$, then there is a real number $\alpha = \Sigma_{n=k} 2^{-4n}$ between 0 and the real number corresponding to $x$. The subset of $M$ which corresponds to real numbers $< \alpha$ is open and closed in $M$. Thus $\bar{G}$ is totally disconnected.

We shall say that a local group is a local $L$-group (left linear group) if (1) a neighbourhood of the unit element is a neighbourhood of zero in a Banach space and zero is unit element (2) the group multiplication satisfies $xy = y + T_y x$ where $T_y$ is a linear transformation depending on $y$, if $x$ and $y$ are sufficiently small. An $L$-group is a group in which a neighbourhood of the unit element is a local $L$-group.

EXAMPLE 2.2.2. (This example is found also in [1].) We let $(x, y)$ stand for an element in $R_3$, $x \in R$, $y \in R_2$. We define a group multiplication in $R_3$ in the following way: $(x_1, y_1) \cdot (x_2, y_2) = (x_1 + x_2, y_2 + V_{x_2} y_1)$ where $x \to V_x$ is a one-parameter group of unitary linear transformations of $R_2$. In this way $R_3$ becomes an $L$-group $G$. Let $p$ be the smallest number $> 0$ such that $V_p = I$. Since $V_{p/2} = -I$ wee see that $(p/2, y) \cdot (p/2, y) = (p, 0)$ and so $(p, y)$ has no square root for $y \neq 0$ and many square roots for $y = 0$. It is easy to see that $(x, y)$ has a unique square root and lies on a one-parameter subgroup if $x = np$, $n \neq \cdots 1, \pm 3, \pm$ and so the set of elements of the form $z^2$ is dense in the group. It is also easy to see that the $w$-sphere around 0 is uniformly free from subgroups for every $w > 0$, all in contrast to Theorem 2.2.5 above. If we put $H = \{(x,y) \mid x = 0\}$ then $H$ and $G/H$ are both commutative but $G$ is not uniform, in contrast to the theorem on p. 52 in [9].

**2.3. Extensions of local groups.** We shall say that a local group satisfies the general associative law in a neighbourhood $U$ of $e$, if any two association schemes such that the partial products of $x_1 \cdot x_2 \cdots \cdot x_n$ are all in $U$ give the same total product. In a theorem by Malcev (see [11]) a local group is locally isomorphic to a topological group if, and only if, the general associative law holds in some neighbourhood of $e$ in the local group. In the next two theoeems we use additive notation of the group operation. In a local group we define $2^n x$ inductively for natural numbers $n$ to be $2^{n-1}x + 2^{n-1}x$.

THEOREM 2.3.1. *If in a commutative local group G with an invariant metric d (1) to every element y corresponds a unique x with* $2x = y$, *(2)* $2x = y \Rightarrow d(x,e) \leqq d(y,e)$ *and* $2^n x_n = y \Rightarrow d(x_n, e) \to 0$ *as* $n \to \infty$, *then G is locally isomorphic to a topological group.*

PROOF. $y/2$ is well-defined for all $y \in G$ and we can inductively define $y/2^n$ for all natural numbers $n$. Choose $\varepsilon > 0$ such that $x_1 + x_2$ is defined if $d(x_j, e) \leqq 2\varepsilon$. We prove that the general associative law holds in $U = \{x \mid d(x,e) \leqq \varepsilon\}$. If $x_1, x_2$ and $y$ are in $U$ and $y = x_1 + x_2$, then $x_1/2 + x_2/2 = y/2$. For $(x_1/2 + x_2/2) + (x_1/2 + x_2/2) = (x_1/2 + x_2/2) + (x_2/2 + x_1/2) = (x_1/2 + x_2) + x_1/2 = x_1/2 + (x_1/2 + x_2) = x_1 + x_2 = y$ and the uniqueness of $y/2$ gives the result. Repeated use of this law gives that if $y = (x_1 + \cdots) + (x_j + \cdots x_n)$ then $y/2 = (x_1/2 + \cdots) + (x_j/2 + \cdots + x_n/2)$ if the association schemes are the same and all partial sums of $y = (x_1 + \cdots) + (x_j + \cdots + x_n)$ are in $U$. And repeated use of this law gives that if $y = (x_1 + \cdots) + (x_j + \cdots + x_n)$ then for natural numbers $m$ $y/2^m = (x_1/2^m + \cdots) + (x_j/2^m + \cdots + x_n/2^m)$ if the association schemes are the same and all partial sums of $y = (x_1 + \cdots) + (x_j + \cdots + x_n)$ are in $U$. Now since $x_1/2^m \to 0$ as $n \to \infty$ we have that if $y_1 = (x_1 + \cdots) + (x_{j_1} + \cdots + x_n)$ and $y_2 = (x_1 + \cdots) + (x_{j_2} + \cdots + x_n)$ where all partial sums of the two elements are in $U$ then $y_1/2^m = y_2/2^m$ if $m$ is sufficiently large. Thus $y_1 = y_2$ and the general associative law holds in U. Malcev's theorem completes the proof.

We shall say that a local group is a local Banach space if it is locally isomorphic to the additive group of some Banach space.

THEOREM 2.3.2. *Let $G$ be a locally complete, commutative, local group with an invariant metric $d$. If for every $y \in G$ there is a unique $x \in G$ such that $y = 2x$ and $2d \cdot (x, e) = d(y, e)$ then $G$ is a local Banach space.*

PROOF. Choose an $\varepsilon > 0$ such that $x_1 + x_2$ is defined if $x_j \in U = \{x \mid d(x,e) \leqq \varepsilon\}$. Then for every element $y \in U$ the element $\alpha y$ is well defined if $|\alpha| \leqq 1$. We form the family $B$ of equivalence classes of pairs $(\alpha, y)$ where $\alpha$ is a real number and $y \in U$. $(\alpha, y)$ and $(\beta, z)$ are said to be equivalent if $(\alpha/K)y = (\beta/K)z \in U$ for some $K$. Then there is an obvious one-to-one correspondence between $U$ and a subset of $B$. We define a metric in $B$ by putting $d_1(\alpha y, \beta z) = \lim_{K\to\infty} K \cdot d((\alpha/K)y, (\beta/K)z)$ where the right-hand side is defined if $K$ is sufficiently large. The value of $d_1$ is obviously independent of the choices of members in the equivalence classes. It is easy to verify that $d_1$ is a metric which coincides with $d$ on $U$. We define multiplication of $\alpha y \in B$ by a real number $\beta$ to be $(\beta\alpha)y$. And then we define addition in $B$ by putting $\alpha y + \beta z = \lim K \cdot ((\alpha/K)y + (\beta/K)z)$. If $x_1, x_2$ and $x_1 + x_2$ are in $U$ this definition coincides with the addition in $G$. The associative law in $B$ is easily verified and so $B$ is a Banach space which is locally isomorphic to $G$. The theorem is proved.

THEOREM 2.3.3. *A local L-group is locally isomorphic to an L-group.*

PROOF. Choose an $\varepsilon > 0$ such that $x_1 x_2$ is defined if $x_j \in U = \{x \mid d(x,e) \leqq \varepsilon\}$. Now if all partial products of $(x_1 \cdots)(x_j \cdots x_n)$ are in $U$, then it is easy to verify that the product equals $x_n + T_{x_n} x_{n-1} + T_{x_n} T_{x_{n-1}} x_{n-2} + \cdots + T_{x_n} T_{x_{n-1}} \cdots T_{x_2} x_1$ and this expression is obviously independent of the association scheme. Malcev's theorem completes the proof.

**2.4. Embeddings of topological groups in linear spaces**. It is a general problem to characterise subgroups of the additive groups of linear spaces in terms of topological or metric groups In this section we give one theorem of this type.

THEOREM 2.4.1. *If in a commutative topological group $G$ there is an invariant metric $d$ such that $d(2x, e) = 2 \cdot d(x, e)$ for all $x \in G$, then $G$ is isomorphic to a subgroup of a Banach space*

PROOF. Since $d(2^m x, e) = 2^m \cdot d(x, e)$ we immediately get $d(nx, e) = |n| \cdot d(x, e)$ for integers $n$ and thus, if for a rational number $r$, $rx$ is defined, we have $d(rx, e) = |r| \cdot d(x, e)$ We embed $G$ in a space $B$ by considering equivalence classes of formal products $rx$ where $r$ is a rational number and $x \in G$ We shall say that two formal products $r_1x$ and $r_2y$ are equivalent if there is a natural number $p$ such that $(pr_1)x$ and $(pr_2)y$ denote the same element of $G$ For those rational numbers $r$ and $x \in G$ where $rx$ is an element of $G$ the formal product thus coincides with this element We define a metric $d_1$ in $B$ by putting $d_1(rx,e) = |r| \cdot d(x,e)$ This definition is obviously independent of the choice of member of the equivalence class and $d_1$ coincides with $d$ on $G$ We define addition in $B$ by letting $(p/q)x + (p_1/q_1)y$ be the equivalence class which contains $(1/qq_1)(pq_1x + p_1qy)$. This definition of addition is consistent with the addition in $G$ The completion of $B$ now obviously is a Banach space which contains $G$ as a subgroup

## 3. Local convexity in groups and differential calculus in locally bounded linear spaces.

**3.1. Characterisation of classes of linear spaces in terms of topological groups.** We recall that a neighbourhood of 0 in a topological linear space is said to be balanced if $x \in U \Rightarrow \alpha x \in U$ if $|\alpha| \leqq 1$ We recall that the family of balanced neighbourhoods of 0 is a base for the neighbourhood system at 0 in a topological linear space And we also recall the definition that a neighbourhood $U$ of 0 in a topological linear space is bounded if for every neighbourhhod $V$ of 0 there is a real $\alpha$ such that $U \subset \alpha V$ We now prove that this definition coincides with our definition of "uniformly dissipative"

THEOREM 3.1.1. *For a neighbourhood $U$ of 0 in a topological linear space the properties (1) and (2) are equivalent*

(1) *For every neighbourhood $V$ of 0 there is a real $\alpha$ such that $U \subset \alpha V$.*

(2) *For every neighbourhood $V$ of 0 there is an integer $n_V$ such that $x \notin V \Rightarrow nx \notin U$ if $n \geqq n_V$*

PROOF. (1) $\Rightarrow$ (2). It is enough to prove that (2) holds for balanced neighbourhoods $V$. If $V$ is a balanced neighbourhood of 0 then (1) gives $U \subset n_V V$ if $n_V \geqq \alpha$ and this gives that $x \notin V \Rightarrow nx \notin n_V V \supset U$ if $n \geqq n_V \cdot$ (2) $\Rightarrow$ (1). If $V$ is a neighbourhood of 0 then (2) gives that $n_V x \in U \Rightarrow x \in V$. Thus $U \subset n_V V$.

We shall say that a topological linear space which has a bounded neighbourhood of 0 is a locally bounded linear space.

LEMMA. *A balanced neighbourhood of 0 in a topological linear space is bounded if and only if it is uniformly free from subgroups.*

PROOF. Since a bounded neighbourhood is uniformly dissipative by Theorem 3.1.1. we have only to show the "if" part. Let $U$ be balanced and uniformly free from subgroups. Thus if $V$ is a balanced neighbourhood of 0 we have $x \notin$ for some $n \leqq n_V$. Since $V$ is balanced this implies $U \subset n_V V$ and so $U$ is bounded. The lemma immediately gives

THEOREM 3.1.2. *A topological linear space is locally bounded if and only if it is uniformly free from small subgroups.*

We now turn to normable spaces.

THEOREM 3.1.3. *A topological linear space is normable if and only if there exist two uniformly dissipative neighbourhoods $U_1$ and $U_2$ of 0 such that, if for $k = 1,2,\cdots,n$, $j = 1,2,\cdots,n$, $kx_j \in U_1$ holds, then $\sum_{j=1}^{n} x_j \in U_2$.*

PROOF. If the topological linear space is normable we can let $U_1$ and $U_2$ both be the unit sphere in some norm. We show the converse. Let $V_1$ be a balanced neighbourhood of 0, $V_1 \subset U_1$. We consider the set $V_2$ which consists of all elements of the form $\sum_{j=1}^{n} x_j$ where $kx_j \in V_1$ for $k,j = 1,2,\cdots\cdots,n$. Then $V_2$ is a balanced neighbourhood of 0. We have by assumption $V_2 \subset U_2$ and so $V_2$ is a bounded neighbourhood of 0. Thus it remains to show only that $\bar{V}_2$ is convex. If $y_1 = \sum_{j=1}^{n} x_{1j}$ and $y_2 = \sum_{k=1}^{m} x_{2k}$ are two elements of $V_2$ then the $2nm$ elements

$$\underbrace{\frac{x_{11}}{2m},\frac{x_{11}}{2m},\cdots,\frac{x_{11}}{2m}}_{m\frac{x_{11}}{2m}:\ s},\frac{x_{12}}{2m},\frac{x_{12}}{2m},\cdots,\frac{x_{12}}{2m},\cdots,\frac{x_{1n}}{2m},\cdots,\frac{x_{1n}}{2m},\underbrace{\frac{x_{21}}{2n},\cdots,\frac{x_{21}}{2n}}_{n\frac{x_{21}}{2n};\ s},\cdots,\frac{x_{2m}}{2n},\cdots,\frac{x_{2m}}{2n}$$

all have the property that when multiplied by a number which does not exceed $2nm$ they remain in $V_1$ since $V_1$ is balanced. And so their sum which is equal to $(y_1 + y_2)/2$ is in $V_2$. Thus $\bar{V}_2$ is convex. The theorem is proved.

The characterisation in Theorem 3.1.3 could be used as a definition of locally convex groups without the assumption of commutativity. However, it seems that also with this condition on a topological group we cannot obtain anything like the pleasant local commutativity property possessed by Lie groups, namely that $d(xyx^{-1}y^{-1},e)$ is much smaller than $\max[d(x,e),d(y,e)]$ if $x$ and $y$ are near $e$. Thus it seems natural instead to lie the condition on a uniformly dissipative group so that the geometrical structure of the group is in some sense similar to the geometrical structure of a Banach space such is our approach in Part 2 of this paper.

**3.2. Spaces with pseudo-norms.** In this and the next section we introduce differential calculus in locally bounded linear spaces. For a survey of what has been done on differential calculus in topological linear spaces the reader is referred to Averbukh and Smolyanov [12]. It seems, however, that our observations have not been made before.

Let $U$ be a bounded symmetric neighbourhood of 0 in the topological linear space $B$. We define a function $|x|$ on $B$ by putting $|x| = \inf 1/|\alpha|$, $\alpha x \in U$ if $x \neq 0$ and $|x| = 0$ if $x = 0$. Then the function $|x|$ has the properties: (1) $|\alpha x| = |\alpha||x|$ for real $\alpha$. (2) $|x| = 0 \Leftrightarrow x = 0$. (3) $|x + y| \leqq K(|x| + |y|)$ for some constant $K$. (4) $|x|$ defines in an obvious way the topology of $B$. (3) holds since the sets $\alpha U$ form a base for the neighbourhood system at 0. Thus $(1/K)U + (1/K)U \subset U$ for some $K$. We shall say that a functional defined on the abstract linear space $A$ is a pseudo-norm on $A$ if it has the properties 1–3 above. We shall say that a topological linear space is a pseudo-normed space if there is a pseudo-norm defined on the space which gives the topology of the space. The class of pseudo-normable topological linear spaces is obviously just the class of topological linear spaces which have a bounded neighbourhood of 0. If $|x|_1$ and $|x|_2$ are two psudo-norms on $A$ which give the same topology on $A$ then $N_1|x|_1 \leqq |x|_2 \leqq N_2|x|_1$ for some positive real numbers $N_1$ and $N_2$. This holds since $|x|_1 \leqq 1$ and $|x|_2 \leqq 1$ are bounded neighbourhoods of 0 in that topology. Now let $B$ and $C$ be pseudo-normed spaces and let $T$ be a continuous linear operator $B \to C$. Put $|T| = \sup |Tx|/|x|$. Then $|T|$ becomes a pseudo-norm for the set of linear operators $B \to C$ and we call the operator topology given by this pseudo-norm the pseudo-norm topology. There is a problem which naturally occurs in the definition of pseudo-norm. If a pseudo-norm is defined as above by a bounded, symmetric and open neighbourhood $U$ of 0 in a locally bounded linear space, then $x \to |x|$ is upper semi-continuous. Does there always exist a neighbourhood $U$ of 0 which makes $x \to |x|$ continuous?

**3.3. Definitions and some properties of derivatives of functions between pseudo-normed spaces.** We follow the terminology of Dieudonné (see [13] Ch. 8). Let $B$ and $C$ be pseudo-normed spaces and let $A$ be an open open subset of $B$. Let $f$ be a function $A \to C$. We shall say that the linear operator $u: B \to C$ is the derivative of $f$ at $x_0$ if $|f(x) - f(x_0) - u(x - x_0)| = o|x - x_0|$ as $x \to x_0$. Now we derive some consequences of this definition.

PROPOSITION 3.3.1. *The derivative of $f$ at $x_0$ is unique.*

PROOF. The proposition follows immediately from the ineqiuality

$$|u_1(x-x_0)-u_2(x-x_0)| \leqq K\cdot(|f(x)-f(x_0)-u_2(x-x_0)|) + K\cdot(|u_1(x-x_0)-f(x)+f(x_0)|).$$

PROPOSITION 3.3.2. *If $f$ is continuous and has the derivative $u$ at $x_0$, then $u$ is continuous.*

PROOF. Since for every $\varepsilon>0$ we have $|f(x)-f(x_0)-u(x-x_0)| \leqq |x-x_0|$ if $x-x_0$ is sufficiently small we have $|u(x-x_0)| \leqq K\cdot(|f(x)-f(x_0)| + \varepsilon|x-x_0|)$ if $x-x_0$ is sufficiently small, which gives that $u$ is continuous.

PROPOSITION 3.3.3. *If $f_1'(x_0)=u_1$ and $f_2'(x_0)=u_2$, then $(f_1+f_2)'(x_0)$ exists and is equal to $u_1+u_2$.*

PROOF. The proposition follows immediately from the inequality

$$|f_1(x)+f_2(x)-f_1(x_0)-f_2(x_0)-u_1(x-x_0)-u_2(x-x_0)| \leqq K\cdot(|f_1(x)-f_1(x_0)-u_1(x-x_0)|+|f_2(x)-f_2(x_0)-u_2(x-x_0)).$$

PROPOSITION 3.3.4. *If $f'(x_0)=u$ then if $\alpha$ is a real number $(\alpha f)'(x_0)$ exists and is equal to $\alpha u$.*

PROPOSITION 3.3.5. *Let $E,F,G$ be three pseudo-normed spaces $A$ an open neighbourhood of $x_0\in E$, $f$ a continuous function of $A$ into $F$, $y_0=f(x_0)$, $B$ an open neighbourhood of $y_0$ in $F$, $g$ a continuous function from $B$ into $G$. Then if $f$ has a derivative at $x_0$ and $g$ has a derivative at $y_0$, the function $h=g\circ f$ has a derivative at $x_0$ and we have $h'(x_0)=g'(y_0)\circ f'(x_0)$.*

PROOF. We have

$$\begin{aligned}&|h(x_0+s)-h(x_0)-(g'(y_0)\circ f'(x_0))\cdot s|\\ &=|g\circ f(x_0+s)-g\circ f(x_0)-(g'(y_0)\circ f'(x_0))\cdot s|\\ &=|g'(y_0)\cdot(f(x_0+s)-f(x_0))+o_1(f(x_0+s)-f(x_0))-(g'(y_0)\circ f'(x_0))\cdot s|\\ &=|g'(y_0)\cdot(f'(x_0)\cdot s+o_2(s))+o_1(f(x_0+s)-f(x_0))-(g'(y_0)\circ f'(x_0))\cdot s|\\ &=|g'(y_0)\cdot o_2(s)+o_1(f(x_0+s)-f(x_0))|=o_3(s).\end{aligned}$$

In the last equality we have applied a variant of Proposition 3.3.2. The proposition is proved.

We do not derive any more properties of this concept of derivative. For normed spaces it is the same as the Frechet derivative. For pseudo-normed spaces it has in its present form one serious defect: there is nothing like a mean-value theorem in its present form and so in order to make Lie theory we have to make the definition of derivative more restrictive. We do not try to solve this problem here, we only give the following example (a similar example is found also in [12]).

EXAMPLE 3.3.1. For $0 < p < 1$ define a pseudo-norm in $L_p(0,1)$ by putting $|f| = (\int_0^1 |f|^p)^{1/p}$. Consider the mapping $t \to f(t)$ $0 \leqq t \leqq 1$ where $f(t)$ is the function in $L_p(0,1)$ which takes the value 1 in the interval $[0,t]$ and the value 0 in the interval $(t,1]$. We have $|f(t) - f(t_0)| = |t - t_0|^{1/p} = o|t - t_0|$ and so $t \to f(t)$ has the derivative 0 for every $t$ without being constant.

STOCKHOLM UNIVERSITY, SWEDEN